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A One-dimensional Cumulus Model Including Pressure Perturbations[1]

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ABSTRACT—A model for shallow cumulus convection is formulated in which the vertical momentum equation and horizontal divergence equation are combined to produce a diagnostic equation for the perturbation pressure field. These equations, together with the first law of thermodynamics and equation of continuity of water substance, are averaged horizontally over the cloud area (a cylinder of constant radius). The resulting set can be integrated in time to study the life cycle of a nonprecipitating cumulus initiated by release of a small buoyant element. The results indicate that the perturbation pressure field plays an essential role by reducing the extremely sharp gradients in velocity near the cloud top, which are common to most other one-dimensional models. Inclusion of the pressure field also makes it possible to predict the radial scale at which the maximum cloud growth rate will occur.

1. INTRODUCTION

In the past several years, considerable effort has been directed toward the numerical simulation of cumulus clouds as well as the parameterization of the effects of cumulus clouds on the large-scale motions. The present status of cumulus modeling and parameterization has recently been reviewed by Ogura (1973). Therefore, previous work is discussed only briefly here.

Although a few cumulus simulations have involved two-dimensional numerical models, most of the modeling work to date has been concentrated on one-dimensional models in which only the vertical variation of the cloud structure is analyzed. This emphasis on one-dimensional modeling has been partly a matter of computational speed and efficiency. But it is also justified by the fact that the vertical distribution of the various fields is of primary importance both for understanding the microphysical processes within the cloud and modeling the effect of the cloud on the larger scale motions.

The various one-dimensional models that have been described in the literature to date may be divided into two classes: quasi-Lagrangian bubble or plume models (e.g., Simpson and Wiggert 1969) and time-dependent Eulerian models (e.g., Ogura and Takahashi 1971). The former type involves upward integration in height following the rise of the bubble or plume. As pointed out by Simpson (1971), the vertical distribution of in-cloud properties given by such a model cannot be interpreted as a vertical profile for a given instant in time but rather should be viewed as giving the cloud properties at each level as the active thermal rises through that level. in this type of model, the entrainment rate into the bubble or plume is a specified parameter of the model and is usually assumed to vary inversely with the cloud radius.

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In the Eulerian models, on the other hand, the vertical momentum equation, thermodynamic energy equation, and the equation of continuity for water substance are integrated numerically to determine the evolution of the various fields as functions of height and time. In these models, the entrainment is generally divided into a dynamic entrainment necessary to satisfy mass continuity requirements and a turbulent entrainment due to lateral mixing by small-scale eddies. The Eulerian models are able to provide vertical profiles of the various fields over the entire life cycle of the cloud. Possibly the most elaborate example to date is the model of Ogura and Takahashi (1971) in which a number of the important microphysical processes are included to simulate the life cycle of a thunderstorm cell.

Despite the great differences among the various one-dimensional models proposed to date, they nearly all share a common defect. This defect is the assumption that the pressure distribution at any point within the cloud is exactly equal to the hydrostatic environmental pressure at the same level. As has been shown by List and Lozowski (1970), however, there must be a substantial pressure perturbation within the cloud. In general, the perturbation vertical pressure gradient force within the cloud will be the same order of magnitude as the buoyancy term in the vertical momentum equation. Some attempts have been made to allow for the perturbation pressure effect by using a "reduced gravity" in the buoyancy term. However, such an empirical procedure cannot model the complete role of the pressure field. In particular, it is the perturbation pressure field that suppressess the growth of clouds as the cloud radius is increased. Thus, explicit inclusion of the pressure field is necessary if the model is to predict the preferred cloud radius.

The purpose of this paper is to demonstrate that the perturbation pressure field can be included in a one-dimensional model, and that the pressure field has a profound effect on the growth of even moderate size cumulus clouds. To keep the calculations as simple as possible, we limit the discussion to shallow nonprecipitating cumuli. All microphysical processes may then be neglected, and the anelastic equations of Ogura and Phillips (1962) may be used to model the dynamics. A somewhat similar study has been carried out by Lee (1971). However, his method of incorporating the pressure perturbation requires more severe assumptions than the present approach.

2. FORMULATION OF THE MODEL

The equations that describe the model are essentially the set used by Ogura (1963) in his two-dimensional simulation of shallow moist convection. However, we replace the horizontal momentum equation by the *divergence equation* obtained by applying the horizontal divergence operator to the horizontal momentum equation. The resulting set of prediction equations may be written in column vector form as:

$$\frac{\partial A}{\partial t}=-\nabla\cdot(\mathbf{V}A)-\frac{\partial}{\partial z}(wA)+S \tag{1}$$

where

$$A=\begin{Bmatrix}\delta\\ w\\ \phi\\ q\end{Bmatrix}$$

and

$$S=\begin{Bmatrix}-\delta^2-c_p\Theta\nabla^2\sigma-\nabla w\cdot\dfrac{\partial \mathbf{V}}{\partial z}-2J(u,v)\\ -c_p\Theta\dfrac{\partial\sigma}{\partial z}+\dfrac{g\theta}{\Theta}-gq_l\\ -w\dfrac{d\phi_0}{dz}\\ -w\dfrac{dq_0}{dz}\end{Bmatrix}.$$

Here δ, w, ϕ, θ, and q designate departures from the environmental values of the horizontal divergence, vertical velocity, specific entropy, potential temperature and water substance mixing ratio, respectively. The pressure is expressed in terms of the nondimensional variable $\sigma\equiv(p/p_0)^{\kappa}$ where p is the perturbation pressure, $p_0=1000$ mb, and $\kappa=R/c_p$. Additional symbols that require definition are Θ, the basic state potential temperature, q_l, the liquid water mixing ratio, ϕ_0, the environmental specific entropy, and q_0, the environmental mixing ratio. All other symbols have their conventional meaning. The specific entropy, ϕ, in the anelastic approximation is defined as

$$\phi=\frac{\theta}{\Theta}+\frac{Lq_v}{c_p\Theta} \tag{2}$$

where L is the latent heat of sublimation and q_v is the departure from the environmental water vapor mixing ratio. Finally, the vertical velocity and horizontal divergence are related by the continuity equation for a Boussinesq fluid,

$$\delta+\frac{\partial w}{\partial z}=0. \tag{3}$$

It is assumed here that the environmental vertical velocity is zero so that the vertical mass transport by the cloud is not compensated locally.

Following the scheme of Asai and Kasahara (1967), we assume that the cloud is a cylindrical column of constant radius and average the equations horizontally over the cloud area. Using the notation of Asai and Kasahara, we define the following fields:

$$\overline{A}=\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a A\,r\,dr\,d\lambda,$$

$$\tilde{A}_a=\frac{1}{2\pi}\int_0^{2\pi}A_a d\lambda \qquad (\text{at } r=a),$$

and

$$A'=A-\overline{A},\qquad A''_a=A_a-\tilde{A}_a. \tag{4}$$

Here, r and λ are the radial and azimuthal coordinates, $r=a$ is the cloud radius, A' denotes a deviation from $\overline{A}$, the horizontal mean for the cloud, and A''_a denotes a deviation from $\tilde{A}_a$, the azimuthal mean at $r=a$. With the aid of eq (4), the horizontal average of eq (1) may be written as

$$\frac{\partial\overline{A}}{\partial t}=-\frac{2}{a}(\tilde{u}_a\tilde{A}_a+\widetilde{u''_aA''_a})-\frac{\partial}{\partial z}(\overline{w}\overline{A}+\overline{w'A'})+\overline{S} \tag{5}$$

where $\widetilde{u''_aA''_a}$ designates the lateral eddy exchange across the cloud boundary and $\overline{w'A'}$ is the vertical eddy flux. To close the system, we must express $\tilde{u}_a$, $\tilde{A}_a$, and the eddy fluxes in terms of the horizontal averaged variables.

Taking the horizontal average of eq (3), we obtain

$$\frac{2}{a}\tilde{u}_a=\bar{\delta}=-\frac{\partial\overline{w}}{\partial z}. \tag{6}$$

Following Asai and Kasahara, we now assume that $\tilde{A}_a$ has the environmental value ($\tilde{A}_a=0$) wherever there is entrainment ($\tilde{u}_a<0$) and has the cloud value ($\tilde{A}_a=\overline{A}$) wherever there is detrainment ($\tilde{u}_a>0$). Thus,

$$-\frac{2}{a}\tilde{u}_a\tilde{A}_a=\begin{cases}\overline{A}\dfrac{\partial\overline{w}}{\partial z}, & u_a>0\\ 0, & u_a<0\end{cases} \tag{7}$$

Next, adapting the eddy exchange hypothesis, we let

$$\widetilde{u''_aA''_a}=\frac{\nu}{a}\overline{A}$$

and

$$\overline{w'A'}=-\nu\frac{\partial\overline{A}}{\partial z} \tag{8}$$

where the eddy exchange coefficient, ν, is defined by $\nu=\alpha^2 a|\overline{w}|$, with $\alpha^2=0.1$ an empirical constant. This representation for the lateral eddy fluxes is identical to the form used by Asai and Kasahara (1967) and Ogura and Takahashi (1971). However, those authors did not explicitly include the vertical eddy flux term; instead, they used an upstream differencing scheme, which produces a computational damping similar to the vertical eddy viscosity used here.

The horizontally averaged divergence equation may now be written, with some approximations, as

$$\frac{\partial\overline{\delta}}{\partial t}=(\tilde{\delta}_a+\overline{\delta})\frac{\partial\overline{w}}{\partial z}-\frac{\partial}{\partial z}(\overline{w}\overline{\delta})+c_p\Theta k^2\overline{\sigma}-\frac{2\nu}{a^2}\overline{\delta}+\frac{\partial}{\partial z}\left(\nu\frac{\partial\overline{\delta}}{\partial z}\right). \quad (9)$$

For simplicity, we have neglected the eddy terms,

$$\overline{\delta'^2},\ \overline{\nabla w'\cdot\frac{\partial\mathbf{V}'}{\partial z}},\ \text{and}\ \overline{2J(u',v')},$$

and we have assumed that the perturbation pressure field varies sufficiently smoothly so that its Laplacian can be approximated by a single normal mode solution with

$$\overline{\nabla^2\sigma}=-k^2\overline{\sigma} \quad (10)$$

where k is a measure of the horizontal scale of the perturbation pressure field.

This representation of the horizontal Laplacian of pressure is the key approximation in the model. Evidence from two-dimensional cloud modeling (Schlesinger 1972) suggests that the pressure perturbation decays exponentially away from the center of the cloud at a rate that depends on the vertical scale of the buoyant thermal. For simplicity in the present calculations, however, we assume that $\sigma(r)$ may be approximated by the first term in a Fourier-Bessel expansion satisfying the conditions that $\partial\sigma/\partial r=0$ at $r=0$ and $\sigma=0$ at $r=a$. Thus, k is the first root of $J_0(ka)=0$, or $k\approx 2.4/a$.

The remaining horizontally averaged prognostic equations are as follows:

$$\frac{\partial\overline{w}}{\partial t}=\tilde{w}_a\frac{\partial\overline{w}}{\partial z}-\frac{\partial}{\partial z}(\overline{w})^2-c_p\Theta\frac{\partial\overline{\sigma}}{\partial z}+\frac{g\overline{\theta}}{\Theta}-g\overline{q}_l-\frac{2\nu}{a^2}\overline{w}+\frac{\partial}{\partial z}\left(\nu\frac{\partial\overline{w}}{\partial z}\right), \quad (11)$$

$$\frac{\partial\overline{\phi}}{\partial t}=\tilde{\phi}_a\frac{\partial\overline{w}}{\partial z}-\frac{\partial}{\partial z}(\overline{w}\overline{\phi})-\overline{w}\frac{d\phi_0}{dz}-\frac{2\nu}{a^2}\overline{\phi}+\frac{\partial}{\partial z}\left(\nu\frac{\partial\overline{\phi}}{\partial z}\right), \quad (12)$$

and

$$\frac{\partial\overline{q}}{\partial t}\tilde{q}_a\frac{\partial\overline{w}}{\partial z}-\frac{\partial}{\partial z}(\overline{w}\overline{q})-\overline{w}\frac{dq_0}{dz}-\frac{2\nu}{a^2}\overline{q}+\frac{\partial}{\partial z}\left(\nu\frac{\partial\overline{q}}{\partial z}\right). \quad (13)$$

With the aid of the continuity equation [eq (6)], we can now combine eq (9) and (11) to obtain the following diagnostic equation for $\overline{\sigma}$:

$$c_p\Theta\left(\frac{\partial^2\overline{\sigma}}{\partial z^2}-k^2\overline{\sigma}\right)=(\tilde{w}_a-\overline{w})\frac{\partial^2\overline{w}}{\partial z^2}+\frac{g}{\Theta}\frac{\partial\overline{\theta}}{\partial z}-g\frac{\partial\overline{q}_l}{\partial z}-2\overline{\delta}^2-\frac{2}{a^2}\frac{\partial\nu}{\partial z}\overline{w}+\frac{\partial}{\partial z}\left(\frac{\partial\nu}{\partial z}\frac{\partial\overline{w}}{\partial z}\right). \quad (14)$$

To complete the system, we require from eq (2) that

$$\overline{\phi}=\frac{\overline{\theta}}{\Theta}+\frac{L\overline{q}_v}{c_p\Theta} \quad (15)$$

where $\overline{q}_v=\overline{q}_{vs}$ for saturated conditions and $\overline{q}_v=\overline{q}$ for unsaturated air. Recalling that $\overline{q}_v$ represents a deviation from the environmental mixing ratio, q_0, we have

$$\overline{q}_{vs}=(q_{vs})_c-q_0=[(q_{vs})_c-(q_{vs})_0]+[(q_{vs})_0-q_o] \quad (16)$$

where $(q_{vs})_c$ and $(q_{vs})_0$ are the saturation mixing ratios for the cloud and the environment, respectively. Following the notation of Ogura (1963), we can write the saturation mixing ratio for shallow convection as

$$q_{vs}=\frac{Re_s(T_{00})}{R_v p_0}\exp\left(\frac{L\theta}{R_v\Theta^2}\right) \quad (17)$$

where R_v is the gas constant for water vapor and $e_s\ (T_{00})$ is the saturation vapor pressure for an adiabatic lapse rate computed from the formula

$$e_s(T_{00})=6.11\times 10^{[7.5(T_{00}-273)/(T_{00}-36)]}. \quad (18)$$

Here

$$T_{00}=T_{\text{ground}}-\frac{g}{c_p}z.$$

Noting that the potential temperature within the cloud is $\overline{\theta}+\theta_0$, we find from eq (16) and (17) that

$$(q_{vs})_0=\frac{Re_s(T_{00})}{R_v p_0}\exp\left(\frac{L\theta_0}{R_v\Theta^2}\right) \quad (19)$$

and

$$\overline{q}_{vs}\approx(q_{vs})_0\left(1+\frac{L\overline{\theta}}{R_v\Theta^2}\right)-q_0. \quad (20)$$

Substituting from eq (20) into (15), we find that for saturated conditions

$$\overline{\theta}_s=\frac{\Theta\phi-\dfrac{L[(q_{vs})_0-q_0]}{c_p}}{1+\dfrac{L^2(q_{vs})_0}{c_p R_v\Theta^2}}, \quad (21)$$

while for unsaturated conditions

$$\overline{\theta}_{\text{uns}}=\Theta\phi-\frac{L\overline{q}}{c_p}. \quad (22)$$

Comparing eq (21) and (22), we see that if $\overline{q}>\overline{q}_{vs}$, then $\overline{\theta}_s>\overline{\theta}_{\text{uns}}$. Thus, at each time step, we compare $\overline{\theta}_s$ and $\overline{\theta}_{\text{uns}}$ to test for saturation. If $\overline{\theta}_s\geq\overline{\theta}_{\text{uns}}$, we set $\overline{\theta}=\overline{\theta}_s$ and let

$$\overline{q}_l=\overline{q}-\overline{q}_{vs}. \quad (23)$$

If $\overline{\theta}_s<\overline{\theta}_{\text{uns}}$, we let $\overline{\theta}=\overline{\theta}_{\text{uns}}$ and set $\overline{q}_l=0$. This treatment of the moist thermodynamic processes is similar to Ogura's (1963) approach. However, the equations have been simplified here by treating all in-cloud fields as departures from known environmental fields. The evolution of a moist convective element of known radius may now be computed by integrating eq (11)–(13) forward in time

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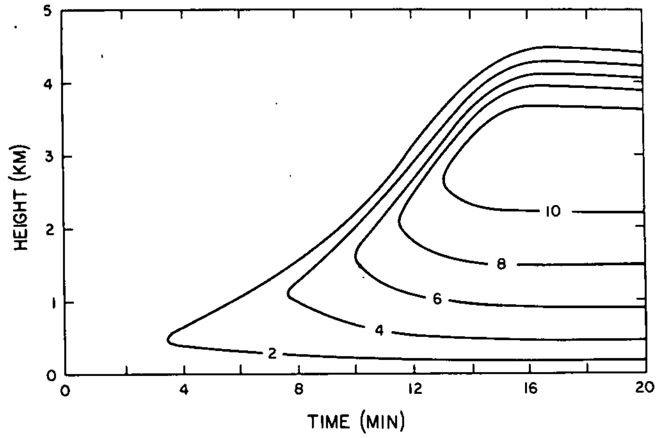

FIGURE 1.—Time-height section of $\bar{w}$ (m/s) for a cloud of radius 0.8 km.

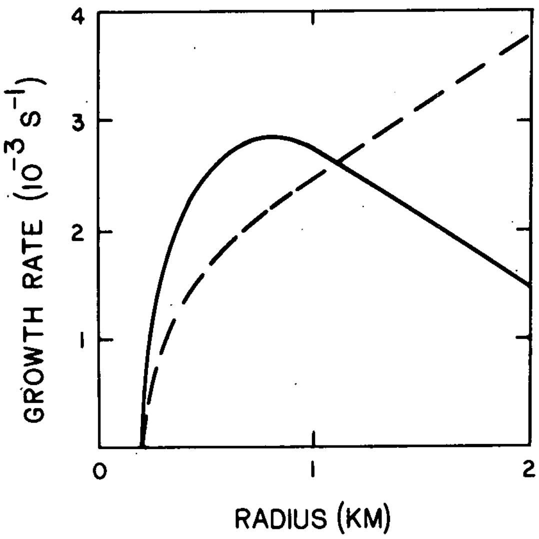

FIGURE 2.—Growth rate of $\bar{w}_{max}$ as a function of cloud radius with the effects of the perturbation pressure included (solid line) and with $\bar{\sigma}=0$ (dashed line).

and using eq (14), (15), and (23) to diagnose $\bar{\sigma}$, $\bar{\theta}$, and $\bar{q}_l$ at each time step. The finite-difference scheme employed in the present model is the two-step Euler backward-difference scheme in time (Haltiner 1971, page 225), and centered differences in space.

3. NUMERICAL RESULTS

Since the primary purpose of this paper is to demonstrate the role of the perturbation pressure field, we have run two parallel sets of numerical experiments; one in which the perturbation pressure effects were included, and a second in which $\bar{\sigma}$ was set to zero. In all of the runs discussed here, environmental conditions are specified as follows:

$$\Theta=300°\text{K},$$

$$p_0=1000 \text{ mb},$$

$$\frac{d\theta_0}{dz}=\begin{cases}3.5°\text{K/km}, & 0<z<3 \text{ km}\\ 9.8°\text{K/km}, & z>3 \text{ km}\end{cases}$$

$$q_0=\beta(q_{vs})_0, \text{ and}$$

$$\beta=1-0.05z \ (z \text{ in km}).$$

The boundary conditions are

$$\bar{w}=\bar{\theta}=\bar{q}=0 \text{ at } z=0, 5 \text{ km},$$

and, for pressure, $\bar{\sigma}=0$ at $z=5$ km and $\partial\bar{\sigma}/\partial z=0$ at $z=0$. As initial conditions, we set $\bar{w}=\bar{q}=0$ and

$$\bar{\theta}=\begin{cases}\delta\theta \sin^2(\pi z/0.6 \text{ km}), & z<0.6 \text{ km}\\ 0, & z>0.6 \text{ km}\end{cases}$$

where $\delta\theta=0.5°$C in most cases. Runs were made for a number of specified cloud radii from $a=0.2$ km to $a=2$ km.

In figure 1, the vertical velocity is shown as a function of time and height for a run in which the cloud radius was 0.8 km and $\delta\theta=0.5°$C. For approximately the first 10 min, the maximum value of the vertical velocity, $\bar{w}_{max}$, grows approximately exponentially. The *growth rate* of $\bar{w}_{max}$ is conveniently defined as the slope of the straight line fitted to a plot of log $\bar{w}_{max}$ versus time during the period of exponential growth. Figure 2 is a plot of the growth rate determined in this manner as a function of cloud radius. For small radii, the growth rates are limited by turbulent entrainment. As the radius increases, the nonhydrostatic part of the perturbation pressure field decreases, and the hydrostatic part tends to balance the buoyancy force. Thus, for cloud radii greater than approximately 1 km, the role of the pressure field is to substantially reduce the growth rate. If the perturbation pressure field is neglected, the growth rate increases linearly with increasing radii because of the reduced role of entrainment. Thus, when the perturbation pressure field is included, we can run the model for several assumed cloud radii to determine the radius for which the growth rate is a maximum. In this manner we can predict the "preferred" radial scale for the cumulus.

Vertical profiles of $\bar{w}$, $\bar{\theta}$, $\bar{\sigma}$, $\bar{q}$, and $\bar{q}_l$ for a cloud of radius 0.8 km are shown in figures 3 and 4 at $t=12$ min during the active development stage. Inclusion of the perturbation pressure field reduces the rate of rise of the thermal and also concentrates the maxima in $\bar{\theta}$, $\bar{q}$, and $\bar{q}_l$ at the head of the rising thermal. The vertical averages of these fields, however, are not changed much by the inclusion of pressure in the model. The primary role of the pressure field in this case is to spread the vertical acceleration field over a larger depth, thus reducing the concentration of $\partial\bar{w}/\partial z$ both above and below the rising thermal. Dynamic entrainment and detrainment near the level of $\bar{w}_{max}$ are thus both decreased, and, hence, the fields of $\bar{\theta}$, $\bar{q}$, and $\bar{q}_l$ may remain more concentrated at that level.

Clearly, for the shallow nonprecipitating cumulus model presented here, the perturbation pressure field does not have a dramatic impact on the cloud profiles for the small radius associated with maximum growth. However, we have demonstrated that inclusion of the pressure field is essential if we wish to predict the preferred radial scale. The pressure field has a profound effect on

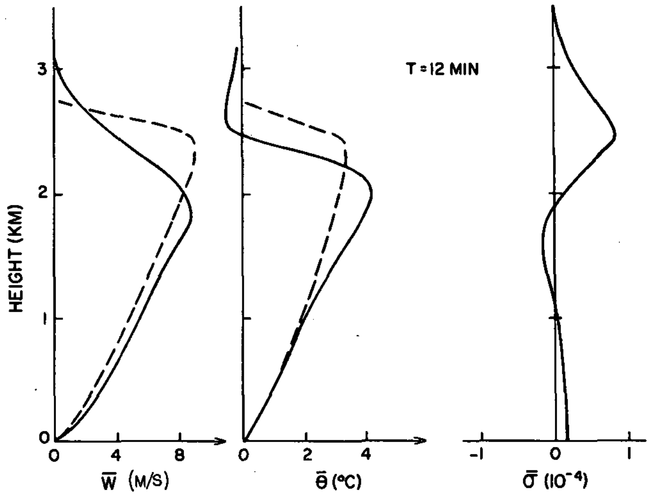

FIGURE 3.—Vertical profiles of $\bar{w}$, $\bar{\theta}$, and $\bar{\sigma}$ at t=12 min for a cloud of radius 0.8 km. Dashed lines are for the case with $\bar{\sigma}=0$; $\bar{\sigma}=10^{-4}$ corresponds to a pressure perturbation of about 0.35 mb.

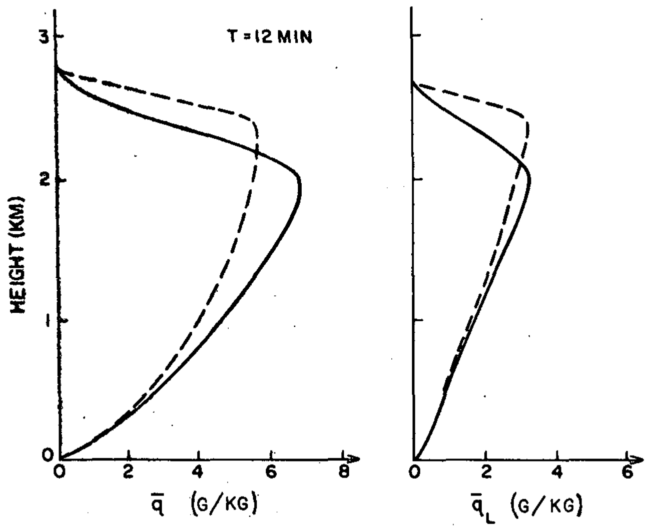

FIGURE 4.—Same as figure 3 for $\bar{q}$ and $\bar{q}_l$.

the growth of larger clouds. For this reason, we expect that the inclusion of perturbation pressure should be important in modeling deep, precipitating cumulonimbus convection. Finally, the vorticity equation might also be horizontally averaged as was done here for the divergence equation. In that case, it should be possible to utilize a one-dimensional model to study the vertical transport of vorticity by the cumulus.

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